Solution to Exercise 7

1. Find all minimum/maximum points of the function

$$F(x, y) = xy - x - y + 3\ ,$$

over the triangle with vertices at $(0, 0), (2, 0)$, and $(0, 4)$.

Solution. $F_x = y - 1; F_y = x - 1$. Therefore, $(1, 1)$ is an interior critical point of F. Let l_1 be the line segment from $(0, 0)$ to $(2, 0)$; l_2 be the line segment from $(2, 0)$ to $(0, 4)$; l_3 be the line segment from $(0, 4)$ to $(0, 0)$.

Along $l_1, F(x, 0) = -x + 3$ is decreasing, possible extrema are $(0, 0)$ and $(2, 0)$. Along $l_2, F(x, -2x + 4) = -2x^2 + 5x - 1, x \in [0, 2]$ has critical point at $x = \dfrac{5}{4}$. It suffices to consider the values of F at $(5/4, 3/2), (2, 0)$ and $(0, 4)$. Along l_3,$F(x, 0) = -y + 3$ is decreasing, it suffices to consider the points $(0, 0)$ and $(0, 4)$.

Finally, we evaluate F at the aforementioned points:

$$F(1, 1) = 2,\ F(0, 0) = 3,\ F(2, 0) = 1,\ F(0, 4) = -1,\ F(5/4, 3/2) = \frac{17}{8}.$$

We see that the maximum point of F is $(0, 0)$ with maximum value $F(0, 0) = 3$ and the minimum point of F is $(0, 4)$ with minimum value $F(0, 4) = -1$.

2. Find all maximum/minimum points of the function

$$z = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ ,$$

in its natural domain.

Solution. The natural domain is $\left\{(x, y) : 1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} \geq 0\right\}$ (the closure of the ellipse) and the function vanishes on its boundary. Computing z_x and z_y,

$$z_x = \frac{y(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2})}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}$$

$$z_y = \frac{x(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2})}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\ .$$

The critical points are $(0, 0), (\pm a/\sqrt{3}, \pm b/\sqrt{3})$ and $(\pm a/\sqrt{3}, \mp b/\sqrt{3})$. At these points:

$$z(0, 0) = 0,\ z\Big(\frac{a}{\sqrt{3}}, -\frac{b}{\sqrt{3}}\Big) = -\frac{ab}{3\sqrt{3}} = z\Big(-\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\Big)\ ,$$

$$z\Big(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\Big) = \frac{ab}{3\sqrt{3}} = z\Big(-\frac{a}{\sqrt{3}}, -\frac{b}{\sqrt{3}}\Big)\ .$$

Therefore, maximum points of z are $(a/\sqrt{3}, b/\sqrt{3})$ and $(-a\sqrt{3}, -b\sqrt{3})$ with maximum value $\dfrac{ab}{3\sqrt{3}}$ and minimum points of z are $(a/\sqrt{3}, -b/\sqrt{3})$ and $(-a\sqrt{3}, b\sqrt{3})$ with minimum value $-\dfrac{ab}{3\sqrt{3}}$.

3. Determine whether the following problems have maximum or minimum in $\mathbb{R}^2$. Not need to find them.

   (a) $g(x,y) = x^3 + y^3 - 3xy$,

   (b) $h(x,y) = x^4 + y^4 - x^2 - xy - y^2$,

   (c) $k(x,y) = \sin xy + x^2$.

   (d) $f(x,y) = (x - y + 1)^2$,

   Suggestion: Theorem 7.2 and its corollaries would be useful.

   Solution. (a) Take $y = 0$, then $g(x,0) = x^3$, which tends to ∞ as x tends to ∞, and tends to $-\infty$ as x tends to $-\infty$. Therefore, g has no maximum nor minimum.

   (b) Since

$$\begin{aligned} x^4 + y^4 &= \frac{1}{2}(x^4 + y^4) + \frac{1}{2}(x^4 + y^4) \\ &\geq \frac{1}{2}(x^4 + y^4) + x^2y^2 \\ &= \frac{1}{2}(x^2 + y^2)^2 , \end{aligned}$$

   and

$$|x^2 + xy + y^2| \leq 2(x^2 + y^2) ,$$

   we have $h(x,y) \to \infty$ uniformly as $x^2 + y^2 \to \infty$, and hence h has no maximum while h has minimum.

   (c) Take $y = 0$, then $k(x,0) = x^2$, which tends to ∞ as x tends to ∞. Therefore, there is no maximum. On the other hand, $\sin xy + x^2 > -1$ (this is clear when $x \neq 0$. When $x = 0$, $k(0,y) = 0 > -1$.) and setting $x_n = 1/n, y_n = -n\pi/2$, $k(x_n, y_n) = -1 + 1/n^2 \to -1$ as $n \to \infty$. Therefore, $\inf k = -1$ but k has no minimum.

   (d) Take $y = 0$, then $f(x,0) = (x+1)^2$, which tends to ∞ as x tends to ∞. Therefore, there does not exist maximum. On the other hand, $f(x,y) = (x - y + 1)^2 \geq 0$ and $= 0$ when $x - y + 1 = 0$. Therefore, all points in the plane $x - y + 1 = 0$ are minimum points.

4. Find all maximum/minimum points of the function $u = x^2 - xy + y^2 - 2x + y$.

   Solution. Note that $2|xy| \leq x^2 + y^2$, $x^2 - xy + y^2 \geq (x^2 + y^2)/2$ so $u \to \infty$ uniformly as $x^2 + y^2 \to \infty$, hence u has no maximum while its minimum exists. Solving

$$u_x = 2x - y - 2 = 0, \quad u_y = -x + 2y + 1 = 0 ,$$

   the only critical point is $(1,0)$, and it must be the minimum.

5. Find all maximum/minmimum points of the function $u = xy^2(1 - x - 2y), \quad x, y > 0$.

   Solution. Consider the triangle with vertices at $(0,0), (1,0)$ and $(0,1/2)$. Note that the slant edge is given by $x + 2y = 1, x \in [0,1]$. Hence we see that $u > 0$ inside the triangle, $u = 0$ on the boundary of the triangle, $u < 0$ outside the triangle. Therefore, u must have maximum inside the triangle, and must be an interior critical point. We have

$$u_x = y^2(1 - x - 2y) - xy^2 , \quad u_y = 2xy(1 - x - 2y) - 2xy^2 .$$

Solving $u_x = u_y = 0$, we have $(x, y) = (1/4, 1/4)$, which is the maximum of u. On the other hand, since $u(x, x) = x^3(1 - 3x) \to -\infty$ as $x \to \infty$, u has no minimum.

6. Let $(x_1, y_1), \cdots, (x_n, y_n)$ be n many order pairs. Find the straight line $y = ax + b$ so that the square error
$$F(a, b) = \sum_{j=1}^{n} (y_j - ax_j - b)^2 \ , \quad (a, b) \in \mathbb{R}^2,$$
is minimized.

Solution. Computing the partial derivatives of F (with notation $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$) :
$$\frac{\partial F}{\partial a} = 2\sum_{j=1}^{n} (y_j - ax_j - b)x_j = 2(x \cdot y - a|x|^2 - b\sum_{j=1}^{n} x_j) \ ,$$
$$\frac{\partial F}{\partial b} = 2\sum_{j=1}^{n} (y_j - ax_j - b)(-1) = -2(\sum_{j=1}^{n} (y_j - ax_j) - nb) \ .$$
Setting $\dfrac{\partial F}{\partial a} = \dfrac{\partial F}{\partial b} = 0$, we have
$$a|x|^2 + b\sum_{j=1}^{n} x_j = x \cdot y,$$
and
$$a\sum_{j=1}^{n} x_j + nb = \sum_{j=1}^{n} y_j \ ,$$
which is a linear system with two variables a, b. Solving the above linear system yields
$$a = \frac{nx \cdot y - (\sum_{j=1}^{n} x_j)(\sum_{j=1}^{n} y_j)}{n|x|^2 - (\sum_{j=1}^{n} x_j)^2} \ ,$$
and in terms of a, b is expressed as
$$b = \frac{1}{n}\Big(\sum_{j=1}^{n} y_j - a\sum_{j=1}^{n} x_j\Big) \ .$$

7. Find all maximum/minimum points of the function
$$w(x, y) = xy + \frac{50}{x} + \frac{20}{y} \ , \quad x, y > 0 \ .$$

Solution. Note that $w \to \infty$ uniformly near the boundary of $\{(x, y) : x, y > 0\}$. Therefore, w has no maximum and has minimum, which must be an interior critical point of w. We have
$$w_x = y - \frac{50}{x^2} \ , \quad w_y = x - \frac{20}{y^2}.$$
Solving $w_x = w_y = 0$, the critical point $(x, y) = (5, 2)$. By the above discussion, $(5, 2)$ is the minimum point of w.

8. Find and classify the critical points of the following functions

   (a) $f_1(x,y) = 9 + 4y - 3x^2 - 2y^2 + 4xy$.

   (b) $f_2(x,y) = 3x - x^3 - 3xy^2$.

   (c) $f_3(x,y) = x^4 + y^4 - 4xy$.

Solution. (a) The only critical point of f_1 is $(2,3)$, and its Hessian matrix is given by $\begin{bmatrix} -6 & 4 \\ 4 & -4 \end{bmatrix}$. The eigenvalues are $\dfrac{-10 \pm \sqrt{68}}{2}$, which are negative. Therefore, $(2,3)$ is a local maximum of f_1.

(b) The critical points of f_2 are $(0,1), (0,-1), (1,0), (-1,0)$, and the Hessian matrix is given by $\begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$.
Substituting each of the above critical points to Hessian matrix and computing the eigenvalues of each of the Hessian matrix, we see that all the critical points are saddle points of f_2.

(c) The critical points of f_3 are $(0,0), (1,1), (-1,-1)$, and the Hessian matrix of f_3 is given by $\begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$.
We see that $(0,0)$ is a saddle point and $(1,1)$ is a local minimum of f_3.

9. Find and classify the critical points of the function

$$H(x,y) = xy\log(x^2+y^2), \quad (x,y) \neq (0,0) ,$$

and $H(0,0) = 0$.

Solution. At $(x,y) \neq (0,0)$,

$$H_x = y\log(x^2+y^2) + \frac{2x^2y}{x^2+y^2}, \quad H_y = x\log(x^2+y^2) + \frac{2xy^2}{x^2+y^2} .$$

On the other hand, by the definition of partial derivatives $H_x(0,0) = H_y(0,0) = 0$. The critical points of H are given by $(0,0), \Big(\pm\dfrac{1}{\sqrt{2e}}, \pm\dfrac{1}{\sqrt{2e}}\Big), \Big(\pm\dfrac{1}{\sqrt{2e}}, \mp\dfrac{1}{\sqrt{2e}}\Big)$.

The Hessian matrix of H is given by

$$\begin{bmatrix} \dfrac{2x^3y + 6xy^3}{(x^2+y^2)^2} & \log(x^2+y^2) + 2 - \dfrac{4x^2y^2}{(x^2+y^2)^2} \\ \log(x^2+y^2) + 2 - \dfrac{4x^2y^2}{(x^2+y^2)^2} & \dfrac{2xy^3 + 6x^3y}{(x^2+y^2)^2} \end{bmatrix} .$$

At $\Big(\dfrac{1}{\sqrt{2e}}, \dfrac{1}{\sqrt{2e}}\Big)$ and $\Big(-\dfrac{1}{\sqrt{2e}}, -\dfrac{1}{\sqrt{2e}}\Big)$, the Hessian becomes

$$\begin{bmatrix} \dfrac{2}{e^2} & 0 \\ 0 & \dfrac{2}{e^2} \end{bmatrix} .$$

At $\left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right), \left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$, the Hessian becomes

$$\begin{bmatrix} -\frac{2}{e^2} & 0 \\ 0 & -\frac{2}{e^2} \end{bmatrix} .$$

Therefore,

$$\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right), \quad \left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right) ,$$

are local minimum points and

$$\left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right), \quad \left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) ,$$

are local maximum points.

It is not clear if the function is twice differentiable at the critical point $(0, 0)$, so we cannot apply the Second Derivative Test. However, H is negative near $(0, 0)$ when $xy > 0$ and positive when $xy < 0$, we see that $(0, 0)$ is a saddle.

10. Find all maximum/minimum points of the function

$$h(x, y, z) = xyz \ ,$$

over the set $\{(x, y, z) : x + y + z = 1, \ x, y, z \geq 0 \ \}$.

Solution. Apply the Lagrange multiplier method to h subject to the constraint $g(x, y, z) = x + y + z - 1$ to get

$$\begin{cases} h_x = \lambda g_x, \\ h_y = \lambda g_y, \\ h_z = \lambda g_z, \\ g(x, y, z) = 0 \ . \end{cases}$$

More explicitly,

$$\begin{cases} yz = \lambda, \\ xz = \lambda, \\ xy = \lambda, \\ x + y + z = 1 \ . \end{cases}$$

Multiplying the first three equations gives

$$x^2 y^2 z^2 = \lambda^3,$$

which implies $xyz = \lambda^{\frac{3}{2}}$. It follows that $x = y = z = \lambda^{\frac{1}{2}}$. Using the last equation we get $x = y = z = 1/3$. Therefore, $(1, 1, 1)/3$ is a maximum point of h with maximum value $1/27$.

On the other hand, h is continuous on the compact set $K = \{(x, y, z) : x + y + z = 1, \ x, y, z \geq 0 \ \}$, $h > 0$ in its interior, and vanishes if and only if $x = 0$ or $y = 0$ or $z = 0$. Therefore, the minimum of h on K must be attained on these boundary points.

11. Consider a hexagon with vertices $(\pm 1, 0), (\pm x, \pm y), x, y \geq 0$, inscribed in the unit circle. Show that the area is maximal when it is a regular hexagon with equal sides and angles.

Solution. By elementary geometry, the area of the hexagon with vertices $(\pm 1, 0), (\pm x, \pm y)$ is given by

$$A(x,y) = 4\Big(xy + \frac{1}{2}y(1-x)\Big) = 4xy + 2y(1-x) = 2xy + 2y \ , \quad 0 \leq x, y \leq 1 \ ,$$

where (x, y) satisfies $x^2 + y^2 = 1$. Therefore, we have to maximize $A(x, y)$ with constraint equation $x^2 + y^2 = 1$, that is, over the compact set $\big\{(x,y) : x^2 + y^2 = 1, x, y \geq 0\big\}$.

By Lagrange multiplier, we have

$$\begin{cases} 2y &= 2\lambda x, \\ 2x + 2 &= 2\lambda y, \\ x^2 + y^2 &= 1 \ . \end{cases}$$

It follows that $y = \lambda x$ and $x + 1 = \lambda^2 y$. Therefore,

$$x = \frac{1}{\lambda^2 - 1} \ , \quad y = \frac{\lambda}{\lambda^2 - 1} \ .$$

Substitute these expressions of x and y to the last equation gives

$$1 + \lambda^2 = (\lambda^2 - 1)^2 \ ,$$

that is,

$$\lambda^4 - 3\lambda^2 = 0 \ .$$

Hence, $\lambda^2 = 0$ or $\lambda^2 = 3$. In the former case, $x = -1$, is excluded. Therefore, $\lambda^2 = 3$, and hence $\lambda = \pm\sqrt{3}$. If $\lambda = -\sqrt{3}$, $y = -\frac{\sqrt{3}}{2}$, is excluded. Therefore, $\lambda = \sqrt{3}$, and hence $(x, y) = (1/2, \sqrt{3}/2)$. Now, A is considered over a compact set and it is equal to 0 at the endpoints $(1, 0), (0, 1)$ and positive inside. So it must attain its maximum. Now $(1/2, \sqrt{3}/2)$ is the only critical point so it must be the maximum of A. You can verify that it gives the regular hexagon by showing that all sides of this hexagon are of equal length.

12. Use Lagrange multiplier to show that the distance from a point z to the hyperplane $H : a \cdot x = b$ is given by

$$\frac{|a \cdot z - b|}{|a|} \ .$$

Solution. Let $x = (x_1, ..., x_n)$ and $p = (p_1, ..., p_n)$ with p being fixed. Let $q(x) = \sum_{j=1}^n (x_j - p_j)^2$ be the square of distance between x and p. We have to minimize $q(x)$ subject to $g(x) = a \cdot x - b = 0$.

Applying the Lagrange multiplier method to $q(x)$ along $g(x) = a \cdot x - b$, we have

$$\begin{cases} 2(x_i - p_i) = \lambda a_i, \quad i = 1, \cdots, n, \\ a \cdot x - b = 0 \ . \end{cases}$$

For each $1 \le i \le n$, multiply a_i on both sides of the first equation and then sum up over all i, we have

$$2\sum_{i=1}^{n}(a_i x_i) - 2\sum_{i=1}^{n}(a_i p_i) = \lambda \sum_{i=1}^{n} a_i^2 \ .$$

We may rewrite it as

$$2b - 2a \cdot p = \lambda |a|^2 \ .$$

Therefore, $\lambda = \dfrac{2b - 2a \cdot p}{|a|^2}$, and

$$\begin{aligned} q(x) &= \sum_{j=1}^{n} \left(\lambda \frac{a_j}{2}\right)^2 \\ &= \left(\frac{2b - 2a\cdot p}{|a|^2}\right)^2 \sum_{j=1}^{n}\left(\frac{a_j}{2}\right)^2 \\ &= \left(\frac{|b - a\cdot p|}{|a|}\right)^2 , \end{aligned}$$

which is the minimum of $q(x)$, i.e. square of the distance from p to the hyperplane. Therefore, the distance is given by $\dfrac{|b - a \cdot p|}{|a|}$.

Note. The distance square function q tends to infinity uniformly so the minimum must attain somewhere. Now there is only one critical point so it must be the minimum point.

13. Find the points on the ellipse $x^2 + xy + y^2 = 3$ that are closest to and farthest from the origin. Hint: Write the equations in the form $ax + by = 0, \ \ cx + dy = 0$, and use the fact that $ad - bc = 0$ if there are non-trivial solutions.

Solution. Let $q(x, y) = x^2 + y^2$ and $g(x, y) = x^2 + xy + y^2 - 3$. We solve the extremal problem for $q(x, y)$ subject to $g(x, y) = 0$. First of all, the problem is over the compact set $\{(x, y) : \ x^2 + xy + y^2 = 3\}$ hence maximum/minimum both are attained somewhere.

Using Lagrange multiplier we have the following system

$$\begin{cases} 2x = \lambda(2x + y), \\ 2y = \lambda(x + 2y), \\ x^2 + xy + y^2 - 3 = 0. \end{cases}$$

The first two equations are

$$\begin{cases} 2(1-\lambda)x - \lambda y = 0, \\ -\lambda x + 2(1-\lambda) y = 0. \end{cases}$$

Since this system has nontrivial solution,

$$\begin{vmatrix} 2(1-\lambda) & -\lambda \\ -\lambda & 2(1-\lambda) \end{vmatrix} = 0 \ ,$$

which is a quadratic equation in λ. Solving it to get two distinct solutions $\lambda_1 = 2$, $\lambda_2 = 2/3$. When $\lambda = 2$, $(x, y) = (\sqrt{3/2}, -\sqrt{3/2})$ or $(-\sqrt{3/2}, \sqrt{3/2})$.

When $\lambda = 2/3$, $(x, y) = (1, 1)$ or $(-1, -1)$. We have

$$q\left(\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\right) = q\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right) = \frac{3}{2}\ ,$$

and

$$q(1,1) = q(-1,-1) = 2\ .$$

Therefore, $(\sqrt{3/2}, -\sqrt{3/2}), (-\sqrt{3/2}, \sqrt{3/2})$ are the points on the ellipse closest to the origin, while $(1,1), (-1,-1)$ are the points on the ellipse farthest from the origin.

14. Let $p \in G$ be a critical point of f subject to $g = 0$. Show that if its Lagrange multiplier is non-zero, then it is also a critical point of g subject to $f - f(p) = 0$.

Solution. As p is a critical point of f under $g = 0$, $\nabla f(p) = \lambda \nabla g(p)$ for some $\lambda \in \mathbb{R}$. Since $\lambda \neq 0$, we have $\nabla g(p) = \mu \nabla f(p)$, where $\mu = \dfrac{1}{\lambda}$. Therefore, p is also a critical point for g under $f - f(p) = 0$.

15. (a) Find the points of the hyperbola $xy = 1$ that are closest to the origin $(0, 0)$.
    (b) Show that the same points maximize xy over the circle $x^2 + y^2 - 2 = 0$.
    (c) Can you explain the "duality" in (a) and (b)?

Solution. (a) Let $h(x, y) = x^2 + y^2$ and $g(x, y) = xy$. We minimize $h(x, y)$ subject to $g(x, y) = 1$. By Lagrange multiplier

$$\begin{cases} 2x = \lambda y, \\ 2y = \lambda x, \\ xy = 1\ . \end{cases}$$

We have $4xy = \lambda^2 xy$. It is easy to see that $x, y \neq 0$, so $\lambda = \pm 2$. When $\lambda = 2$, $x = y$, and hence $(x, y) = (1, 1)$ or $(-1, -1)$. When $\lambda = -2$, $x = -y$, and hence $x^2 = -1$, which is impossible. Therefore, the critical points are $(1, 1)$ and $(-1, -1)$, and the minimum of q is given by $q(1, 1) = q(-1, -1) = 2$. $(1, 1)$ and $(-1, -1)$ are the points on the hyperbola closest to the origin.

(b) Let $g(x, y) = xy$ and $h(x, y) = x^2 + y^2$. We maximize $g(x, y)$ subject to $h(x, y) = 2$. By Lagrange multiplier

$$\begin{cases} y = 2\mu x, \\ x = 2\mu y, \\ x^2 + y^2 = 2. \end{cases}$$

Note that this system is the same as in (a) with $\mu = \dfrac{1}{\lambda}$. Hence $\mu = 1/2$ or $-1/2$, each of which implies $x = \pm y$, and hence $(x, y) = (1, 1), (-1, 1), (1, -1), (-1, -1)$. Since $f(1, 1) = f(-1, -1) = 1$ and $f(1, -1) = f(-1, 1) = -1$, $(1, 1)$ and $(-1, -1)$ are the maximum points of f.

(c) (a) and (b) show that the minimization of h over $g = 1$ and the maximization of g over $h = 2$ have the same extremal points.

16. A company uses the Cobb-Douglas production function

$$N(x, y) = 10x^{0.6}y^{0.4}$$

to estimate a new product. Here x is the number of units of labor and y is the number of units of capital required to produce $N(x, y)$ units of the product. Each unit of labor costs \$30 and each unit of capital costs \$60. If \$300,000 is budgeted for the production, determine how that amount should be allocated to maximize production, and find the maximum production.

Solution. The constraint is $30x + 60y = 300,000$ which is simplified to $x + 2y = 10,000$. We have

$$\begin{cases} 6x^{-0.4}y^{0.4} = \lambda, \\ 4x^{0.6}y^{-0.6} = 2\lambda, \\ x + 2y = 10,000\ . \end{cases}$$

We find $x = 3y$ and then $y = 2,000$, and $x = 6,000$. Therefore, the amount which maximizes the production is $(x, y) = (6,000, 2,000)$ with maximum production $N(6,000, 2,000) = 10(6,000)^{0.6}(2,000)^{0.4}$.

17. Let T be a right triangle with sides x, y and hypotenuse z. Find the one maximizing the area subject to the perimeter constraint $x + y + z = 10$. Does there exist an area minimizing one?

Solution. Let $f(x, y, z) = \frac{1}{2}xy$, $g(x, y, z) = x + y + z$ and $h(x, y, z) = x^2 + y^2 - z^2$ (this constraint asserts this is a right triangle). Applying the Lagrange multiplier method to $f(x, y, z)$ subject to $g(x, y, z) = 10$ and $h(x, y, z) = 0$, we have the following system of equations:

$$\begin{cases} \frac{1}{2}y = \lambda + 2\mu x, \\ \frac{1}{2}x = \lambda + 2\mu y, \\ 0 = \lambda - 2\mu z, \\ x + y + z = 10, \\ x^2 + y^2 = z^2\ . \end{cases}$$

We have $(1 + 4\mu)(x - y) = 0$. If $\mu = -\frac{1}{4}$,

$$\frac{1}{2}y = \lambda - \frac{1}{2}x\ , \quad 0 = \lambda + \frac{1}{2}z,$$

together imply $x + y + z = 0$, which is impossible. Therefore, $(1 + 4\mu)(x - y) = 0$ implies $x = y$. Then

$$\begin{cases} 2x + z = 10, \\ 2x^2 = z^2, \end{cases}$$

which is readily solved to give $x = 10 \pm 5\sqrt{2}$. Since x cannot exceed 10, only $10 - 5\sqrt{2}$ is admitted. From it we find $z = 10\sqrt{2} - 10$. We conclude that the triangle maximizing the area under these two constraints has sides $x = y = 10 - 5\sqrt{2}$ and $z = 10\sqrt{2} - 10$. The minimum of area is 0 when the triangle collapses into a line segment, so it cannot be attained.

18. Find the maximum/minimum points of the function $g(x,y,z) = xy + z^2$ subject to the constraints $y - x = 0$ and $x^2 + y^2 + z^2 = 4$.

Solution. Let $h(x,y,z) = y - x$ and $k(x,y,z) = x^2 + y^2 + z^2$. We have

$$\begin{cases} y = -\lambda + 2\mu x, \\ x = \lambda + 2\mu y, \\ 2z = 2\mu z, \\ y - x = 0, \\ x^2 + y^2 + z^2 = 4\ . \end{cases}$$

We have $x = y$ and then $\lambda = 0$. And $2z(1-\mu) = 0$. If $\mu = 1$, then $x = y = 0$ and $z = \pm 2$. Therefore, $(x,y,z) = (0,0,\pm 2)$ is a critical point of g. If $z = 0$, then $2x^2 = 4$, i.e. $x = \pm\sqrt{2}$. Therefore, $(x,y,z) = (\sqrt{2},\sqrt{2},0), (-\sqrt{2},-\sqrt{2},0)$ are critical points of g. Since $g(0,0,\pm 2) = 4$ and $g(\sqrt{2},\sqrt{2},0) = g(-\sqrt{2},-\sqrt{2},0) = 2$, $(0,0,\pm 2)$ are maximum points while $(\sqrt{2},\sqrt{2},0), (-\sqrt{2},-\sqrt{2},0)$ are minimum points.

Note. The underlying set is the intersection of the unit sphere with a plane hence it is compact.